THE COMPLETE SPLITTING NUMBER OF A LASSOED LINK

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Abstract. In this paper, we define a lassoing on a link, a local addition of a trivial knot to a link. Let $K$ be an $s$-component link with the Conway polynomial non-zero. Let $L$ be a link which is obtained from $K$ by $r$-iterated lassoings. The complete splitting number $\text{split}(L)$ is greater than or equal to $r + s - 1$, and less than or equal to $r + \text{split}(K)$. In particular, we obtain from a knot by $r$-iterated component-lassoings an algebraically completely splittable link $L$ with $\text{split}(L) = r$. Moreover, we construct a link $L$ whose unlinking number is greater than $\text{split}(L)$.

1. Introduction

The splittability of a link is one of the basic concepts in knot theory. For example, the splittability interacts with polynomial invariants: the Alexander polynomial and the Conway polynomial have zero values for a splittable link. Jones polynomial and skein polynomial have formulae with respect to the split sum. Moreover, the splittabilities of links or spatial graphs are studied and applied to other subjects: chemistry, biology, psychology, etc. For example, Kawauchi proposed a model of prion proteins as a spatial graph [3], and Yoshida studied its splittability which concerns with the study of prion diseases: mad cow disease, scrapie, Creutzfeldt-Jakob disease, etc. [8]. Another example is about a model of human mind which is also proposed by Kawauchi [2], [3]; by considering one’s mind as a knot and by considering a mind relation of $n$ persons as an $n$-component link, the models “mind knots” and “mind links” are studied. The splittability of a link corresponds to the “self-releasability” of a mind link.

For a two-component link, Adams defined the splitting number which represents how distant the link is from a splittable link [1]. In this paper, we define for an $n$-component link $L$ ($n = 2, 3, 4, \ldots$) the complete splitting number $\text{split}(L)$ which represents how distant the link is from a completely splittable link. The unlinking number $u(L)$ of a link $L$ is the minimal number of crossing changes in any diagram of $L$ which are needed to obtain the trivial link $L$. Since a trivial link is completely
splittable, we have \( \text{split}(L) \leq u(L) \). Lassoing is a crossing-changing and loop-adding local move as shown in Figure 1 (we give the precise definitions of completely splittable, complete splitting number, and a lassoing in Section 2).

For any \( r \)-component link \( L = L_1 \cup L_2 \cup \ldots \cup L_r \) \((r = 1, 2, 3, \ldots)\) with the Conway polynomial \( \nabla(L) \neq 0 \), there are \( \ell \)-iterated lassoings from \( L \) to an algebraically completely splittable link \( L^* \) with \( \nabla(L^*) \neq 0 \) where \( \ell = \sum_{i<j} |\text{Link}(L_i, L_j)| \) (we define an algebraically completely splittable link in Section 2). For any \( s \)-component link \( K = K_1 \cup K_2 \cup \ldots \cup K_s \) \((s \geq 1)\) with \( \nabla(K) \neq 0 \), there are \((\ell+u)\)-iterated lassoings from \( K \) to an algebraically completely splittable link \( L \) with trivial components such that \( \nabla(L) \neq 0 \) where \( \ell = \sum_{i<j} |\text{Link}(K_i, K_j)| \) and \( u = \sum_{i=1}^{s} u(K_i) \). In this paper, we show the following theorem:

**Theorem 1.1.** Any link \( L \) obtained from any \( s \)-component link \( K = K_1 \cup K_2 \cup \ldots \cup K_s \) \((s = 1, 2, 3, \ldots)\) by \( r \)-iterated lassoings \((r = 0, 1, 2, \ldots)\) satisfies

\[ r + \text{split}(K) \geq \text{split}(L) \geq r + s - 1. \]

We have the following corollaries:

**Corollary 1.2.** For any \( s \)-component link \( K = K_1 \cup K_2 \cup \ldots \cup K_s \) \((s = 1, 2, 3, \ldots)\) with \( \text{split}(K) = s - 1 \), and any integer \( r \geq \ell + u \) where \( \ell = \sum_{i<j} |\text{Link}(K_i, K_j)| \) and \( u = \sum_{i=1}^{s} u(K_i) \), there are \( r \)-iterated lassoings from \( K \) to an algebraically completely splittable link \( L \) with trivial components such that \( \text{split}(L) = r + s - 1. \)

**Corollary 1.3.** Let \( K \) be a knot. Let \( L \) be a link which is obtained from \( K \) by \( r \)-iterated lassoings \((r = 1, 2, 3, \ldots)\). Then \( L \) has \( \text{split}(L) = r \).

We define a *component-lassoing* to be the lassoing at a self-crossing point of a diagram. We have the following corollary:
Corollary 1.4. Every link $L$ obtained from a knot $K$ by $r$-iterated component-lassoings ($r = 1, 2, 3, \ldots$) is an $(r+1)-$component algebraically completely splittable link with $\text{split}(L) = r$.

For example, the link $7_6^2$ depicted in Figure 2 which is a link obtained from a trefoil knot by a single component-lassoing, has the linking number zero and $\text{split}(7_6^2) = 1$. We also remark that $u(7_6^2) = 2$ ([6]). Adams also showed in [1] that there is a two-component link, each component of which is trivial, but such that its splitting number is less than its unlinking number, like the link $7_6^2$. We show in Section 5 that for any integer $r > 0$ and any knot $K$ with Nakanishi’s index $e(K) > 2r$, any link $L$ obtained from $K$ by $r$-iterated lassoings is a link such that $\text{split}(L) < u(L)$, i.e., $L$ is non-trivial by any $r$ crossing changes.

2. Complete splitting number

Let $L = L_1 \cup L_2 \cup \cdots \cup L_r$ be a link consisting of sublinks $L_i$ ($i = 1, 2, \ldots, r$). A link $L$ is splittable into $L_1, L_2, \ldots, L_r$ if there exist mutually disjoint 3-balls $B_i$ ($i = 1, 2, \ldots, r$) in $S^3$ such that $L_i \subset B_i$. For example, the link $M$ in Figure 3 is splittable into $M_1$ and $M_2$ whereas the link $N$ is not splittable into $N_1$ and $N_2$. A link $L$ is splittable if $L$ is splittable into subdiagrams $L_1$ and $L_2$, where $L = L_1 \cup L_2$, $L_1, L_2 \neq \phi$. For example, the link $M$ in Figure 3 is a splittable link.

A link $L$ is completely splittable if $L$ is splittable into all the knot components of $L$. In particular, a knot is assumed as a non-splittable link but a completely splittable
A link $L$ is \textit{algebraically completely splittable} if every two knot components $K_i$ and $K_j$ of $L$ have the linking number $\text{Link}(K_i, K_j) = 0$. For example, the link $E$ in the left hand of Figure 4 is not completely splittable but algebraically it is completely splittable.

![Figure 4](image)

The \textit{complete splitting number} $\text{split}(D)$ of a link diagram $D$ is the minimal number of crossing changes which are needed to obtain a diagram of a completely splittable link. For example, the link diagram $F$ in the right hand in Figure 4 has $\text{split}(F) = 1$. As a relation to the warp-linking degree $\text{ld}(D)$ of $D$, we have $\text{split}(D) \leq \text{ld}(D)$, where the warp-linking degree is a restricted warping degree which can be calculated directly or by using matrices [7]. The \textit{complete splitting number} $\text{split}(L)$ of a link $L$ is the minimal number of crossing changes in any diagram of the link which are needed to obtain a completely splittable link.

Let $p$ be a crossing point of a link diagram $D$. We put a lasso around $p$, i.e., we apply a crossing change at $p$, and add a loop alternately around the crossing as shown in Figure 1. Then, we obtain another link diagram $D'$. The diagram $D'$ is said to be obtained from $D$ by \textit{lassoing} $p$. Let $L'$ be the link which has the diagram $D'$. The link $L'$ is said to be obtained from $L$ by a \textit{lassoing}. For example, we obtain the Borromean ring from the Hopf link by a lassoing (see Figure 5).

![Figure 5](image)

A link $L'$ is said to be obtained from $L$ by \textit{$r$-iterated lassoings} if $L'$ is obtained from $L$ by lassoings $r$ times iteratively. For example, the link $L$ in Figure 6 is a link obtained from a trivial knot by two-iterated lassoings. Since a lassoing depends on the choice of a crossing point and the choice of a diagram of the link, we may have many types of link by a lassoing.
3. Conway polynomial

Let $\nabla(L; z)$ be the Conway polynomial of a link $L$ with an orientation. We have the following lemma:

**Lemma 3.1.** We have

\[
\nabla \left( \begin{array}{c}
\begin{array}{c}
\text{lasso-} \\
\end{array}
\end{array} z \right) = -z^3 \nabla \left( \begin{array}{c}
\begin{array}{c}
\text{lasso-} \\
\end{array}
\end{array} z \right),
\nabla \left( \begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array} z \right) = -z^3 \nabla \left( \begin{array}{c}
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\end{array} z \right),
\nabla \left( \begin{array}{c}
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\end{array} z \right) = z^3 \nabla \left( \begin{array}{c}
\begin{array}{c}
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\end{array} z \right),
\nabla \left( \begin{array}{c}
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\end{array} z \right) = z^3 \nabla \left( \begin{array}{c}
\begin{array}{c}
\text{lasso-} \\
\end{array}
\end{array} z \right).
\]

Proof. We obtain the first equality by the skein relations in Figure 7. The other equalities are similarly obtained. \qed

**Example 3.2.** The link diagram $D$ in Figure 8 is obtained from a diagram of a trefoil knot by 2-iterated lassoings. Then we have $\nabla(L) = z^3 \times z^3 \times \nabla(3_1) = z^6(1 + z^2)$, where $L$ is a link represented by $D$, and $3_1$ is a trefoil knot.

We remark that for a link $L'$ with $\nabla(L') = 0$, there are no lassoings from $L'$ to $L$ with $\nabla(L) \neq 0$. We have the following corollary:

**Corollary 3.3.** Let $L$ be a link obtained from a link $L'$ with $\nabla(L') \neq 0$, in particular from a knot $K$, by $r$-iterated lassoings ($r = 1, 2, 3, \ldots$). Then we have $\nabla(L) \neq 0$. 
In this section, we prove Theorem 1.1. Before the proof, we define some notions which are due to [4] to prove Theorem 1.1. For the integral Laurent polynomial ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, a multiplicative set of $\Lambda$ is a subset $S \subseteq \Lambda \setminus \{0\}$ which satisfies the following three conditions: the units $\pm t^i$ ($i \in \mathbb{Z}$) are in $S$, the product $gg'$ of any elements $g$ and $g'$ of $S$ is in $S$, and every prime factor of any element $g \in S$ is in $S$. For the quotient field $Q(\Lambda)$ of $\Lambda$ and a multiplicative set $S$ of $\Lambda$, $\Lambda_S = \{f/g \in Q(\Lambda) | f \in \Lambda, g \in S\}$ is a subring of $Q(\Lambda)$. For a finitely generated $\Lambda$-module $H$,
$H_S$ be the $\Lambda_S$-module $H \otimes_\Lambda \Lambda_S$, and $e_S(H)$ the minimal number of $\Lambda_S$-generators of $H_S$. We take $e_S(H) = 0$ when $H = 0$. We call $e_S(H)$ the $\Lambda_S$-rank of $H$. Let $L$ be an oriented link in $S^3$, and $E(L) = cl(S^3 - L)$ the compact exterior of $L$. Let $\tilde{E}(L) \to E(L)$ be the infinite cyclic covering which is induced from the epimorphism $\gamma_L : \pi_1(E(L)) \to \mathbb{Z}$ sending each oriented meridian of $L$ to $1 \in \mathbb{Z}$. Then we can regard $H_1(\tilde{E}(L))$ as a finitely $\Lambda$-module. We denote $e_S(H_1(\tilde{E}(L)))$ by $e_S(L)$. Let $L$, $L'$ be links which have the same number of components. By Theorem 2.3 in [4], we immediately have

\begin{equation}
    d^X(L, L') \geq |e_s(L) - e_s(L')|,
\end{equation}

where $d^X(L, L')$ denotes the $X$-distance between $L$ and $L'$. We prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $L$ be a link which is obtained from a link $K = K_1 \cup K_2 \cup \cdots \cup K_s$ with $\nabla(K) \neq 0$ by $r$-iterated lassoings ($r = 1, 2, 3, \ldots$). Let $L'$ be a completely splittable link which is obtained from $L$ by $m$-times crossing changes, where $m = \text{split}(L) = d^X(L, L')$. We set $S = \Lambda - \{0\}$. Since $L'$ is completely splittable and the number of components of $L'$ is $r + s$, we have

\begin{equation}
    e_s(L') = r + s - 1.
\end{equation}

The Alexander polynomial of $L$ is non-zero because the Conway polynomial of $L$ is non-zero by Corollary 3.3. Hence we have

\begin{equation}
    e_s(L) = 0.
\end{equation}

By substituting the equalities (2), (3) and $d^X(L, L') = \text{split}(L)$ into the inequality (1), we have

\begin{equation}
    \text{split}(L) \geq r + s - 1.
\end{equation}

And from the $r$-iterated lassoings, we have

\begin{equation}
    r + \text{split}(K) \geq \text{split}(L).
\end{equation}

Hence we have the inequality

\begin{equation}
    r + \text{split}(K) \geq \text{split}(L) \geq r + s - 1.
\end{equation}

As the contraposition of Theorem 1.1, we have the following corollary:

**Corollary 4.1.** Let $K = K_1 \cup K_2 \cup \cdots \cup K_s$ be an $s$-component link. If $K$ has $\text{split}(K) < s - 1$, then $\nabla(K) = 0$. 
5. Non-triviality

In this section, we discuss the non-trivialities of completely splittable links which are obtained from $L$ in Corollary 1.3 by $r$ crossing changes ($r = 1, 2, \ldots$). For a link $L$ obtained from a knot $K$ by $r$-iterated lassoings, we have the following theorem:

**Theorem 5.1.** If a link $L$ is obtained from a knot $K$ with $e(K) > 2r$ by $r$-iterated lassoings ($r = 1, 2, 3, \ldots$), then we have $\text{split}(L) = r$ and $u(L) > r$. Further, if the $r$-iterated lassoings are all component-lassoings, then $L$ is an algebraically completely splittable link.

Proof. Let $L_0 = K_1 + K_2 + \cdots + K_{r+1}$ be a completely splittable link which is obtained from $L$ by $r$ crossing changes. For the integral Laurent polynomial ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, we take the multiplicative set $S$ of $\Lambda$ so that $S$ is the set of units of $\Lambda$. Then $e_S(L)$ is equivalent to Nakanishi’s index $e(L)$ [4]. Since we can consider $L_0 = K_1 + K_2 + \cdots + K_{r+1}$ to be a connected sum $O^{r+1} \# K_1 \# K_2 \# \ldots \# K_{r+1}$, we have

$$H_1(\tilde{E}(L_0)) \cong H_1(\tilde{E}(O^{r+1})) \oplus H_1(\tilde{E}(K_1 \# K_2 \# \ldots \# K_{r+1})) \cong \Lambda^r \oplus H_1(\tilde{E}(K_1 \# K_2 \# \ldots \# K_{r+1})).$$

And by [5], we have $e(L_0) = r + e(K_1 \# K_2 \# \ldots \# K_{r+1})$. By substituting this into the inequality (1), we have

$$d^X(L, L_0) \geq |e(L) - e(L_0)| \geq e(L) - r - e(K_1 \# K_2 \# \ldots \# K_{r+1}).$$

Recall that $d^X(L, L_0) = \text{split}(L) = r$. Then we have

(4) \hspace{1cm} r \geq e(L) - r - e(K_1 \# K_2 \# \ldots \# K_{r+1}).

Next, we consider another completely splittable link $K + O^r$ which is obtained from $L$ by the $r$ anti-lassoings (see Figure 9).

![Figure 9](anti-lassoing.png)

Since $K + O^r = O^{r+1} \# K$, we have

$$H_1(\tilde{E}(L_0)) \cong \Lambda^r \oplus H_1(\tilde{E}(K)).$$
And by [5], we have $e(K + O^r) = r + e(K)$. Hence we have

$$r \geq r + e(K) - e(L)$$

by [4]. By summing the inequalities (4) and (5), we have

$$2r \geq e(K) - e(K_1 \# K_2 \# \ldots \# K_{r+1}).$$

Therefore we have

$$u(L_0) \geq u(K_1 \# K_2 \# \ldots \# K_{r+1}) \geq e(K_1 \# K_2 \# \ldots \# K_{r+1}) \geq e(K) - 2r.$$

Hence $L_0$ is non-trivial if $e(K) > 2r$. □

By taking a knot which has Nakanishi’s index large enough, we can construct a link such that the unlinking number is greater than the complete splitting number. Here is an example.

Example 5.2. Since the knot $K$ in Figure 10 which is the connected sum of $2r + 1$ trefoil knots has Nakanishi’s index $e(K) = 2r + 1$, any link $L$ obtained from $K$ by $r$-iterated lassoings has the unlinking number more than $r$ whereas split$(L) = r$.

![Figure 10](image)

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