Volumes of orientable 2-cusped hyperbolic 3-manifolds

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Abstract

Let M be an orientable 2-cusped hyperbolic 3-manifold. We show that $\mathrm{vol}(M) > 2.3952v_0$, where v_0 is the volume of the regular ideal tetrahedron which is $1.01494\ldots$

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1 Introduction.

C. Adams showed the following theorem in [1].

THEOREM **1.1** *Let M be an n-cusped hyperbolic 3-manifold, namely a complete hyperbolic 3-manifold of finite volume with n cusps. Then, we have*

$$\mathrm{vol}(M) \geq nv_0$$

where v_0 is the volume of the regular ideal tetrahedron which is $1.01494\ldots$

For $n = 1, 2$, we can actually obtain non-orientable n-cusped hyperbolic 3-manifolds with $\mathrm{vol}(M) = nv_0$, by glueing the faces of n regular ideal tetrahedra.

As for orientable cusped hyperbolic 3-manifolds, C. Cao and R. Meyerhoff showed the following theorem in [4].

THEOREM **1.2** *The minimum volume of orientable 1-cusped hyperbolic 3-manifolds is* $2v_0$.

It is generally believed that the Whitehead link complement, whose volume is $3.66382\ldots$, has the minimum volume among orientable 2-cusped hyperbolic 3-manifolds. In this paper, we prove the following Theorem 1,3.

THEOREM **1.3** *Let M be an orientable 2-cusped hyperbolic 3-manifold. Then, we have*

$$\mathrm{vol}(M) > 2.3952v_0.$$

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2 Cusp volume.

We denote the cusps of M by c_1 and c_2. First, we expand the horospherical section of c_1 until it collides with itself, and denote it by C_1. Next we expand the horospherical section of c_2 until it collides with either C_1 or itself, and denote it by C_2. It is easy to get for $i = 1, 2$,

$$\mathrm{vol}(C_i) = \frac{1}{2}\mathrm{area}(\partial C_i).$$

The following inequality is shown in [1];

$$\mathrm{vol}(M) \geq \frac{2}{\sqrt{3}}(\mathrm{vol}(C_1) + \mathrm{vol}(C_2))v_0.$$

Thus to prove Theorem 1.3, we will show

$$\mathrm{area}(\partial C_1) + \mathrm{area}(\partial C_2) > 2.3952\sqrt{3}.$$

We use the upper-half-space model $\mathbf{H}^3$ of the hyperbolic 3-space, and identify the boundary of $\mathbf{H}^3$ with $\mathbf{C} \cup \{\infty\}$ and the group of orientation preserving isometries of $\mathbf{H}^3$ with $\mathrm{PSL}(2, \mathbf{C})$. Let Γ be the discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$ such that $M = \mathbf{H}^3/\Gamma$. For a horoball H centered at x, we denote $\{\gamma \in \Gamma | \gamma(H) = H\}$ by Γ_H or Γ_x.

To estimate the area of ∂C_1 and ∂C_2, we need the following Lemma 2.1 and 2.2.

LEMMA **2.1** *Let H be a horoball which is a lift of C_i $(i = 1.2)$ in $\mathbf{H}^3$. Suppose j horoballs $H^1, \ldots, H^j$ touch H, and $H^k \cap \gamma(H^l) = \emptyset$ or $\{\text{one point}\}$ for $\gamma \in \Gamma_H \backslash \{id\}$. Then*

$$\mathrm{area}(\partial C_i) \geq \frac{\sqrt{3}}{2}j.$$

PROOF OF LEMMA 2.1. We can assume the center of H is ∞ and its boundary is the Euclidean plane at height 1. By projecting the images of $H^1, \ldots, H^j$ by Γ_H on ∂H, we get a circle packing on ∂H. It is well known that amoung circle packings on the Euclidean plane, the hexagonal packing is the densest one, with density $\pi/2\sqrt{3}$. By the assumption, we get j disks $D_1, \ldots, D_j$ with radius 1/2 on ∂C_i such that $int(D_k) \cap int(D_l) = \emptyset$, where $int(D_k)$ is the set of innerpoints of D_k. Thus $\Sigma_{k=1}^{j}\mathrm{area}(D_k)/\mathrm{area}(\partial C_i) \leq \pi/2\sqrt{3}$. Therefore

$$\mathrm{area}(\partial C_i) \geq \frac{\sqrt{3}}{2}j.$$

LEMMA **2.2** *Let H be a horoball which is a lift of C_i. If $\alpha \in \Gamma - \Gamma_H$, then $\alpha(H)$ is not transformed to $\alpha^{-1}(H)$ by any element of Γ_H.*

PROOF OF LEMMA 2.2. Suppose $\gamma_0\alpha(H) = \alpha^{-1}(H)$ for some $\gamma_0 \in \Gamma_H$. Then $\gamma_0\alpha\gamma_0(H) = \gamma_0\alpha(H) = \alpha^{-1}(H)$ and hence $(\gamma_0\alpha)^2(\alpha^{-1}(H)) = \gamma_0\alpha\gamma_0(H) = \alpha^{-1}(H)$. On the other hand we have $(\gamma_0\alpha)^2(H) = \gamma_0\alpha\alpha^{-1}(H) = \gamma_0(H) = H$. Thus $(\gamma_0\alpha)^2$ fixes the centers of $\alpha^{-1}(H)$ and H, and hence $(\gamma_0\alpha)^2$ is identity. Then so is $\gamma_0\alpha$ as Γ is torsion free, and $\gamma_0 = \alpha^{-1} \notin \Gamma_H$, which contradicts the assumption.

3 Proof of Theorem 1.3.

Let H_∞ be the horoball centered at ∞, whose boundary is the Euclidean plane at height 1. By taking a conjugate of Γ, we assume that H_∞ is a lift of C_1. Furthermore we assume that 0 corresponds to c_2 and that a lift of C_2 centered at 0, which we denote by H_0, has the largest Euclidean height in the lifts of C_2.

As C_1 collides with itself, there is a horoball H^1 which is a lift of C_1 such that H^1 touches H_∞. Since H_∞ and H^1 are lifts of C_1, there exists $\alpha_0 \in \Gamma$ such that $H^1 = \alpha_0(H_\infty)$. By Lemma 2.2, the horoball $H^1 = \alpha_0(H_\infty)$ is not transformed to $\alpha_0^{-1}(H_\infty)$ by any element of Γ_∞.

Case 1. Suppose C_2 collides with itself. We consider the area of ∂C_1. We expand H_0 until it collides with H_∞, and name it H'. The horoball H' does not intersect with any horoball which is a lift of C_1.

Therefore we have three horoballs H', $\alpha_0(H_\infty)$ and $\alpha_0^{-1}(H_\infty)$ which touch H_∞, and which are not transformed to one another by any element of Γ_∞. By Lemma 2.1, we have $\mathrm{area}(\partial C_1) \geq 3\sqrt{3}/2$. As C_2 collides with itself, we also have $\mathrm{area}(\partial C_2) \geq 3\sqrt{3}/2$. Thus we get

$$\mathrm{area}(\partial C_1) + \mathrm{area}(\partial C_2) \geq 3\sqrt{3}.$$

Theorem 1.3 follows immediately.

Case 2. Suppose C_2 collides with C_1. Thus the Euclidean height of H_0 is 1. As three horoballs H_0, $\alpha_0(H_\infty)$ and $\alpha_0^{-1}(H_\infty)$, which touch H_∞, are not transformed to one another by any element of Γ_∞, by Lemma 2.1, we have

$$\mathrm{area}(\partial C_1) \geq \frac{3\sqrt{3}}{2}.$$

Put $\iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, ${}^\iota\gamma = \iota\gamma\iota^{-1}$ for $\gamma \in \mathrm{PSL}_2(\mathbf{C})$ and ${}^\iota G = \iota G\iota^{-1}$ for a subgroup G of $\mathrm{PSL}_2(\mathbf{C})$. The horoball $\iota(H_0)$ is a lift of C_2 centered at ∞ whose boundary is the Euclidean plane with height 1, and $\iota(H_\infty)$ is a lift of C_1 with Euclidean height 1 centered at 0. On $\partial\iota(H_0)$, the Euclidean structure is induced from the hyperbolic structure of $\mathbf{H}^3$. Put $\{p\} = \iota(H_\infty) \cap \iota(H_0)$. Let us denote by $d(\gamma)$ the Euclidean distance between p and $\gamma(p)$ on $\partial\iota(H_0)$ for $\gamma \in {}^\iota(\Gamma_0)$, and choose ${}^\iota\sigma \in {}^\iota(\Gamma_0)$ such that

$$d({}^\iota\sigma) = \min\{d(\gamma) | \gamma \in {}^\iota(\Gamma_0)\backslash\{id\}\}.$$

We denote $d({}^\iota\sigma)$ simply by r. The center of $\iota(H_0)$ is ∞ and its boundary is the Euclidean plane at height 1. Hence $\iota(H_\infty) \cap {}^\iota\sigma\iota(H_\infty) = \emptyset$ or one point. We have $r \geq 1$. The Euclidean height of $\sigma(H_\infty)$ and $\sigma^{-1}(H_\infty)$ is $1/r^2$.

REMARK. The horoball H_0 touches $\sigma(H_\infty)$, and the Euclidean distance between the centers of these horoballs is $1/r$. Thus the Euclidean distance between the centers of $\sigma(H_0)$ and the horoball whose Euclidean height is 1, is larger than or equal to $1/r$.

As M is an orientable cusped hyperbolic 3-manifold, ${}^\iota(\Gamma_0)$ is generated by two parabolic elements. We can choose ${}^\iota\tau \in {}^\iota(\Gamma_0)$ such that ${}^\iota(\Gamma_0) = \langle {}^\iota\sigma, {}^\iota\tau \rangle$ and $\pi/3 \leq \angle {}^\iota\sigma(p)p{}^\iota\tau(p) \leq \pi/2$ on $\partial\iota(H_0)$. Thus, we have

$$\mathrm{area}(\partial C_2) \geq \frac{\sqrt{3}r^2}{2}.$$

We get

$$\mathrm{area}(\partial C_1) + \mathrm{area}(\partial C_2) \geq \frac{3\sqrt{3}}{2} + \frac{\sqrt{3}r^2}{2}.$$

If $r \geq 1.3381$,

$$\frac{3\sqrt{3}}{2} + \frac{\sqrt{3}r^2}{2} \geq 2.39525\sqrt{3}.$$

Theorem 1.3 follows immediately for $r \geq 1.3381$. Hence, we suppose $1 \leq r < 1.3381$.

Case 2.1. Suppose $1 < r < 1.3381$. We consider $\mathrm{area}(\partial C_1)$.

By Lemma 2.2, $\sigma(H_\infty)$ is not transformed to $\sigma^{-1}(H_\infty)$ by any element of Γ_∞. As C_1 collides with itself, H_∞ touches $H^1 = \alpha_0(H_\infty)$, $\alpha_0^{-1}(H_\infty)$ and H_0, which are not transformed to one another by any element of Γ_∞.

Because $r \neq 1$, the Euclidean height of the horoballs $\sigma(H_\infty)$ and $\sigma^{-1}(H_\infty)$ are less than 1. Therefore we have five horoballs $\sigma(H_\infty)$, $\sigma^{-1}(H_\infty)$, H_0, $\alpha_0(H_\infty)$ and $\alpha_0^{-1}(H_\infty)$ which are not transformed to one another by any element of Γ_∞.

We consider the Euclidean distance of the centers of the horoballs. The Euclidean distance of the centers of the horoballs with Euclidean height 1 is larger than or equal to 1, that of the horoballs with Euclidean height 1 and $1/r^2$ is larger than or equal to $1/r$, and that of the horoballs with the Euclidean height $1/r^2$ is larger than or equal to $1/r^2$. Let D_1, D_2, D_3 be disks on $\partial\mathbf{H}^3 = \mathbf{C}\cup\{\infty\}$ with radius $1/2$ centered at the centers of H_0, $\alpha_0^{-1}(H_\infty)$, $\alpha_0(H_\infty)$ respectively, and D_4, D_5 disks on $\partial\mathbf{H}^3 = \mathbf{C}\cup\{\infty\}$ with radius $1/r - 1/2$

centered at the centers of $\sigma(H_\infty)$, $\sigma^{-1}(H_\infty)$ respectively. Since $1/r^2 \geq 2(1/r - 1/2)$, these disks are not transformed to one other by any element of Γ_∞. Thus there exist five disks on area(∂C_1) which are disjoint. To estimate area(∂C_1), we will show the following lemma in Section 4.

LEMMA **3.1** *Suppose a Euclidean torus T contains three disks D_1, D_2, D_3 with radius $1/2$ and two disks D_4, D_5 with radius $0.11 < a \leq 0.5$ such that $int(D_i) \cap int(D_j) = \emptyset$ for $i \neq j$. Then the following inequality holds;*

$$\mathrm{area}(T) \geq \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 12\tan\frac{\pi}{6}a^2.$$

By the above Lemma 3.1, we have

$$\mathrm{area}(\partial C_1) \geq \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 12\tan\frac{\pi}{6}(\frac{1}{r} - \frac{1}{2})^2.$$

Hence

$$\mathrm{area}(\partial C_1) + \mathrm{area}(\partial C_2) \geq \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 12\tan\frac{\pi}{6}(\frac{1}{r} - \frac{1}{2})^2 + \frac{\sqrt{3}r^2}{2}.$$

It is not hard to check the right side of the inequality is larger than $4.4796\ldots > 2.3952\sqrt{3} = 4.1486\ldots$ for $1 < r \leq 1.3381$.

Case 2.2. Suppose $r = 1$. Recall that we choose ${}^\iota\tau \in {}^\iota(\Gamma_0)$ such that ${}^\iota(\Gamma_0) = \langle {}^\iota\sigma, {}^\iota\tau \rangle$ and $\pi/3 \leq \angle {}^\iota\sigma(p)p{}^\iota\tau(p) \leq \pi/2$ on $\partial\iota(H_0)$. Put $s = d({}^\iota\tau)$. We have

$$\mathrm{area}(\partial C_2) \geq \sqrt{s^2 - \frac{1}{4}}.$$

As area$(\partial C_1) \geq 3\sqrt{3}/2$, if $s \geq 1.63$, we get

$$\mathrm{area}(\partial C_1) + \mathrm{area}(\partial C_2) \geq 4.1494\ldots \geq 2.3952\sqrt{3} = 4.1486\ldots$$

Therefore we may assume $s < 1.63$.

We will show $\sigma(H_\infty)$, $\sigma^{-1}(H_\infty)$, $\tau(H_\infty)$, $\tau^{-1}(H_\infty)$ are not transformed to one another by any element of Γ_∞. Suppose $\sigma(H_\infty) = \gamma\tau(H_\infty)$ for some $\gamma \in \Gamma_\infty$. (Other cases are proved by the same way.) Then $\gamma\sigma^{-1}\gamma\tau(H_\infty) = \gamma\sigma^{-1}\sigma(H_\infty) = \gamma(H_\infty) = H_\infty$. The centers of ${}^\iota\sigma\iota(H_\infty)$, ${}^\iota\sigma^{-1}\iota(H_\infty)$, ${}^\iota\tau\iota(H_\infty)$, ${}^\iota\tau^{-1}\iota(H_\infty)$ are the vertices of a parallelogram and the distance of 0 and each of these is 1. Thus the centers of $\sigma(H_\infty)$, $\sigma^{-1}(H_\infty)$, $\tau(H_\infty)$, $\tau^{-1}(H_\infty)$ are also the vertices of a parallelogram. As $\sigma(H_\infty) = \gamma\tau(H_\infty)$ and $\gamma \in \Gamma_\infty$, we get $\gamma\sigma^{-1}(H_\infty) = \tau^{-1}(H_\infty)$. Hence $\gamma\sigma^{-1}\gamma\tau(\tau^{-1}(H_\infty)) = \gamma\sigma^{-1}\gamma(H_\infty) = \gamma\sigma^{-1}(H_\infty) = \tau^{-1}(H_\infty)$. Thus $\gamma\sigma^{-1}\gamma\tau$ fixes H_∞ and $\tau(H_\infty)$. As Γ is torsion free, $\gamma\sigma^{-1}\gamma\tau$ is identity. We get $0 = \gamma\sigma^{-1}\gamma\tau(0) = \gamma\sigma^{-1}\gamma(0)$. Since $\gamma\sigma^{-1}\gamma\tau$ is identity, $\gamma\tau\gamma\sigma^{-1}$ is also identity and

$0 = \gamma\tau\gamma\sigma^{-1}(0) = \gamma\tau\gamma(0)$. Thus we have $\gamma\sigma^{-1}\gamma(0) = \gamma\tau\gamma(0)$. Hence $\tau\sigma^{-1}(\gamma(0)) = \gamma(0)$. Since the fixed point of $\tau\sigma^{-1}$ is 0, we obtain $\gamma(0) = 0$. It contradicts the assumption that $\gamma \in \Gamma_\infty$.

We consider five horoballs H_0, $\sigma(H_\infty)$, $\sigma^{-1}(H_\infty)$, $\tau(H_\infty)$, and $\tau^{-1}(H_\infty)$. Three horoball H_0, $\sigma(H_\infty)$, $\sigma^{-1}(H_\infty)$ have Euclidean height 1, and two horoballs $\tau(H_\infty)$ and $\tau^{-1}(H_\infty)$ have the Euclidean height $1/s^2$.

In the next section, we prove the following proposition.

PROPOSITION 3.2 *Suppose a Euclidean torus T contains three disks D_1, D_2, D_3 with radius $1/2$ and two disks D_4, D_5 with radius $0.11 < a \leq 0.5$ such that $int(D_i) \cap int(D_j) = \emptyset$ for $i \neq j$. Then*

$$\operatorname{area}(T) \geq \begin{cases} \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 8a^2\tan\frac{\pi}{4} & 0.11 \leq a < a_1 \\ \frac{21}{4}\tan\frac{\pi}{7} + 4a^2\tan\frac{\pi}{4} + 5a^2\tan\frac{\pi}{5} & a_1 \leq a < a_2 \\ \frac{7}{2}\tan\frac{\pi}{7} + \frac{3}{2}\tan\frac{\pi}{6} + 10a^2\tan\frac{\pi}{5} & a_2 \leq a < a_3 \\ \frac{3}{4}\tan\frac{\pi}{6} + 12a^2\tan\frac{\pi}{6} & a_3 \leq a \leq 0.5 \end{cases}$$

where

$$a_1 = \frac{1}{2}\sqrt{\frac{7\tan\pi/7 - 8\tan\pi/8}{4\tan\pi/4 - 5\tan\pi/5}},\ a_2 = \frac{1}{2}\sqrt{\frac{6\tan\pi/6 - 7\tan\pi/7}{4\tan\pi/4 - 5\tan\pi/5}},\ a_3 = \frac{1}{2}\sqrt{\frac{6\tan\pi/6 - 7\tan\pi/7}{5\tan\pi/5 - 6\tan\pi/6}}.$$

By the above proposition, we get

$$\operatorname{area}(\partial C_1) \geq \begin{cases} \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 8(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{4} & \frac{2}{2a_1+1} < s < \frac{100}{61} = 1.6393\ldots \\ \frac{21}{4}\tan\frac{\pi}{7} + 4(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{4} + 5(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{5} & \frac{2}{2a_2+1} < s \leq \frac{2}{2a_1+1} \\ \frac{7}{2}\tan\frac{\pi}{7} + \frac{3}{2}\tan\frac{\pi}{6} + 10(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{5} & \frac{2}{2a_3+1} < s \leq \frac{2}{2a_2+1} \\ \frac{3}{4}\tan\frac{\pi}{6} + 12(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{6} & 1 \leq s \leq \frac{2}{2a_3+1} \end{cases}$$

Put

$$A(s) = \begin{cases} \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 8(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{4} + \sqrt{s^2 - \frac{1}{4}} & \frac{2}{2a_1+1} < s < \frac{100}{61} = 1.6393\ldots \\ \frac{21}{4}\tan\frac{\pi}{7} + 4(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{4} + 5(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{5} + \sqrt{s^2 - \frac{1}{4}} & \frac{2}{2a_2+1} < s \leq \frac{2}{2a_1+1} \\ \frac{7}{2}\tan\frac{\pi}{7} + \frac{3}{2}\tan\frac{\pi}{6} + 10(\frac{1}{s} - \frac{1}{2})^2\tan\frac{\pi}{5} + \sqrt{s^2 - \frac{1}{4}} & \frac{2}{2a_3+1} < s \leq \frac{2}{2a_2+1} \\ \frac{3}{4}\tan\frac{\pi}{6} + 12\tan\frac{\pi}{6}(\frac{1}{s} - \frac{1}{2})^2 + \sqrt{s^2 - \frac{1}{4}} & 1 \leq s \leq \frac{2}{2a_3+1} \end{cases}.$$

Then we have

$$\operatorname{area}(\partial C_1) + \operatorname{area}(\partial C_2) \geq A(s).$$

For $1 \leq s \leq 2/(2a_3 + 1)$,

$$A'(s) \ = \ \frac{s}{\sqrt{s^2 - \frac{1}{4}}} - \frac{12\sqrt{3}}{s^2}(\frac{1}{s} - \frac{1}{2})$$

$$\begin{aligned} &\leq \frac{2/(2a_3+1)}{\sqrt{(2/(2a_3+1))^2 - \frac{1}{4}}} - \frac{12\sqrt{3}}{1}\left(\frac{1}{1} - \frac{1}{2}\right) \\ &= -9.2861\ldots \end{aligned}$$

By the same fashion,

$$A'(s) \leq -3.0207\ldots$$

for $2/(2a_3+1) < s \leq 2/(2a_2+1)$, and

$$A'(s) \leq -1.1042\ldots$$

for $2/(2a_2+1) < s \leq 2/(2a_1+1)$. Thus $A(s)$ is a decreasing function for $1 \leq s \leq 2/(2a_1+1)$. It is not hard to see for $\frac{2}{(2a_1+1)} < s < 100/61 = 1.6393\ldots$

$$A(s) \geq A(1.5280\ldots) = 4.14866\ldots > 2.3952\sqrt{3} = 4.14861\ldots$$

Therefore we obtain

$$\text{area}(\partial C_1) + \text{area}(\partial C_2) > 2.3952\sqrt{3}.$$

4 Proof of Proposition 3.2.

To prove Proposition 3.2, we use the following Lemma 4.1.

LEMMA **4.1** *Let n, m be integers such that $m - 2 \geq n > 2$. Then we have*

$$n \tan\frac{\pi}{n} + m \tan\frac{\pi}{m} \geq (n+1)\tan\frac{\pi}{n+1} + (m-1)\tan\frac{\pi}{m-1}.$$

PROOF OF LEMMA 4.1. Put $g(x) = x\tan\frac{\pi}{x}$ and $f(x) = g(x+1) - g(x)$. Then

$$g''(x) = \frac{2\pi\tan\frac{\pi}{x}}{x^3\cos^2\frac{\pi}{x}} > 0$$

for $x > 2$. We have $f'(x) = g'(x+1) - g'(x) > 0$ for $x > 2$. Thus we get $f(m-1) \geq f(n)$. Hence

$$m\tan\frac{\pi}{m} - (m-1)\tan\frac{\pi}{m-1} \geq (n+1)\tan\frac{\pi}{n+1} - n\tan\frac{\pi}{n}.$$

We obtain

$$n \tan\frac{\pi}{n} + m \tan\frac{\pi}{m} \geq (n+1)\tan\frac{\pi}{n+1} + (m-1)\tan\frac{\pi}{m-1}.$$

PROOF OF LEMMA 3.2. Take the universal cover of T, which is a Euclidean plane $\mathbf{E}^2$. Let $\tilde{D}_i$ be a lift of D_i. Put

$$H(\tilde{D}_i, \gamma(\tilde{D}_j)) = \{x \in \mathbf{E}^2 : d(x, \tilde{D}_i) \leq d(x, \gamma(\tilde{D}_j)\}$$

for $\gamma \in \pi_1(T)$, and

$$\tilde{S}_i = \bigcap_{1 \le j \le 5, \gamma \in \pi_1(T)} H(\tilde{D}_i, \gamma(\tilde{D}_j)).$$

Then $\tilde{S}_i$ is a polygon which contains $\tilde{D}_i$. Let $S_1, \ldots, S_5$ be the covering projection images of $\tilde{S}_1, \ldots, \tilde{S}_5$ respectively. By $S_1, \ldots, S_5$, we get a cell decomposition of T.

Let V, E and F be the number of vertices, edges, and faces of this cell decomposition of T respectively, and V_n the numbers of vertices with order n. Then

$$\begin{aligned} E &= \sum_{n \ge 3} \frac{nV_n}{2} \\ &\ge \frac{3}{2} \sum_{n \ge 3} V_n \\ &= \frac{3}{2} V. \end{aligned}$$

We have

$$V \le \frac{2}{3} E.$$

By the Euler characteristic number, we get

$$V - E + F = 0.$$

Since the number of faces F is 5, we get

$$E \le 15.$$

We denote the number of edges of S_i by n_i. Then

$$\sum_{i=1}^{5} n_i = 2E.$$

Hence

$$\sum_{i=1}^{5} n_i \le 30.$$

Suppose $n_4 + n_5 \ge 12$. We will show $\mathrm{area}(S_i) \ge 2\sqrt{3}a^2$ for $i = 4, 5$, where $2\sqrt{3}a^2$ is obviously equal to the area of the regular hexagon which circumscribes to ∂D_i. (See [5].)

The following proposition is proved in [5].

PROPOSITION 4.2 *Let C be a circle, P a regular n-gon which circumscribes to C, and U a circle which circumscribes to P. If an n-gon S contains C, then* $\mathrm{area}(S \cap U) \ge \mathrm{area}(P)$.

By Proposition 4.2, if an n-gon P contains a circle C, the area of P is larger than or equal to the area of the regular n-gon which circumscribes to C. The area of the regular n-gon is $a^2 \tan \pi/n$ where a is the radios of C. Thus we can assume $n_i \geq 7$. Let P_i be the regular hexagon which circumscribes to $\partial D_i (i = 4, 5)$ and U_i the circle which circumscribes to P_i. We will show area$(S_i \cap U_i) \geq$ area$(P_i) = 2\sqrt{3}a^2$. Let O_j be the center of $D_j (j = 1, \ldots, 5)$, and D_l and D_k the disks corresponding to the adjacent edges of S_i, say e_l and e_k respectively. Drop a perpendicular line from O_i to e_k and e_l. Put the feets of these perpendicular lines M and N respectively. We will show that the distance M and N is larger than or equal to a. Let M' (resp. N') be a point such that of $O_i M' : M' O_k = a : 1/2$(resp. $O_i N' : N' O_l = a : 1/2$). We have

$$MN \geq M'N'.$$

Figure 1.

By the assumption, we get $O_k O_l \geq 2a$. By the definition of M' and N', $M'N' : O_k O_l = a : a + 1/2$. Therefore we have $M'N' \geq 2a^2/(1/2 + a)$. We can easily see $M'N' \geq a$. Hence we can obtain

$$MN \geq a.$$

The distance between the feets of the perpendiculars is larger than or equal to a. Let A_i be the annulus which is bounded by ∂D_i and U_i. There exists at most 7 points on A_i such that each distance between the points is larger than or equal to a.

Suppose the number of the feets of the perpendiculars is less than or equal to 6. Then there exists a 6-gon S' such that area$(S_i \cap U_i) =$ area$(S' \cap U_i)$ and S' contains D_i.

Figure 2.

By Proposition 4.2, area$(S' \cap U_i) \geq$ area(P_i), and hence area$(S_i \cap U_i) \geq$ area(P_i).

Suppose the number of the feets of the perpendiculars is 7. It is not hard to check the length of perpendicular line from O_i to an edge is larger than $1.13831a$ when each distance of feets of these perpendicular is larger than or equal to a.

Figure 3.

Let S'' be the regular 7-gon, whose center is O_i and the foot of perpendicular line to an edge is $1.13831a$. Then area$(S_i \cap U) >$ area$(S'' \cap U_i)$. We can get area$(S'' \cap U_i) > 4.1582a^2$. Thus the area$(P_i \cap U) > 4.1582a^2 > 2\sqrt{3}a^2$.

Figure 4.

As $n_4 + n_5 \geq 12$, we have $n_1 + n_2 + n_3 \leq 18$. Let P_i be the regular n_i-gon which circumscribes to $\partial D_i (i = 1, \ldots, 3)$. Since S_i contains D_i, we have $\text{area}(S_i) \geq \text{area}(P_i)$.

$$\begin{aligned} \sum_{i=1}^{3} \text{area}(S_i) &\geq \sum_{i=1}^{3} \text{area}(P_i) \\ &= \sum_{i=1}^{3} \frac{n_i}{4} \tan \frac{\pi}{n_i}. \end{aligned}$$

We will estimate the right side of the above inequality. As $x \tan \pi/x$ is a decreasing function for $x > 2$, we can assume $\sum_{i=1}^{3} n_i = 18$. By Lemma 4.1,

$$\sum_{i=1}^{3} \frac{n_i}{4} \tan \frac{\pi}{n_i} \geq \sum_{i=1}^{3} \frac{6}{4} \tan \frac{\pi}{6} = \frac{3\sqrt{3}}{2}.$$

Hence

$$\sum_{i=1}^{5} \text{area}(S_i) \geq \frac{3\sqrt{3}}{2} + 4\sqrt{3}a^2.$$

Put

$$f_1(a) = \frac{3\sqrt{3}}{2} + 4\sqrt{3}a^2.$$

Suppose $n_4 + n_5 \leq 11$.

$$\begin{aligned} \sum_{i=1}^{5} \text{area}(S_i) &\geq \sum_{i=1}^{5} \text{area}(P_i) \\ &= \sum_{i=1}^{3} \frac{n_i}{4} \tan \frac{\pi}{n_i} + \sum_{i=4}^{5} n_i a^2 \tan \frac{\pi}{n_i}. \end{aligned}$$

We may assume $n_1 \leq n_2 \leq n_3$ and $n_4 \leq n_5$. By Lemma 4.1, we consider six cases; $(n_1, n_2, n_3, n_4, n_5) = (6, 6, 7, 5, 6), (6, 7, 7, 5, 5), (7, 7, 7, 4, 5), (7, 7, 8, 4, 4), (7, 8, 8, 3, 4),$ $(8, 8, 8, 3, 3)$. Put

$$f_2(a) = 3 \tan \frac{\pi}{6} + \frac{7}{4} \tan \frac{\pi}{7} + 5a^2 \tan \frac{\pi}{5} + 6a^2 \tan \frac{\pi}{6},$$

$$f_3(a) = \frac{7}{2} \tan \frac{\pi}{7} + \frac{3}{2} \tan \frac{\pi}{6} + 10a^2 \tan \frac{\pi}{5},$$

$$f_4(a) = \frac{21}{4} \tan \frac{\pi}{7} + 4a^2 \tan \frac{\pi}{4} + 5a^2 \tan \frac{\pi}{5},$$

$$f_5(a) = \frac{7}{2} \tan \frac{\pi}{7} + 2 \tan \frac{\pi}{8} + 8a^2 \tan \frac{\pi}{4},$$

$$f_6(a) = \frac{7}{4} \tan \frac{\pi}{7} + 4 \tan \frac{\pi}{8} + 3a^2 \tan \frac{\pi}{3} + 4a^2 \tan \frac{\pi}{4},$$

and

$$f_7(a) = 6 \tan \frac{\pi}{8} + 6a^2 \tan \frac{\pi}{3}.$$

Hence we get

$$\sum_{i=1}^{5} \operatorname{area}(S_i) \geq \min_j f_j(a).$$

As

$$f_7(a) - f_5(a) = (6\tan\frac{\pi}{3} - 8\tan\frac{\pi}{4})a^2 + 4\tan\frac{\pi}{8} - \frac{7}{2}\tan\frac{\pi}{7},$$

we have

$$f_7(a) > f_5(a)$$

for $a > \sqrt{(\frac{7}{4}\tan\frac{\pi}{4} - 2\tan\frac{\pi}{8})/(3\tan\frac{\pi}{3} - 4\tan\frac{\pi}{4})} = 0.1095\ldots$ By the same way, we get

$$f_6(a) > f_5(a)$$

for $a > \sqrt{(\frac{7}{4}\tan\frac{\pi}{4} - 2\tan\frac{\pi}{8})/(3\tan\frac{\pi}{3} - 4\tan\frac{\pi}{4})} = 0.1095\ldots$ We can easily get;

$$f_5(a) < f_4(a) \text{ for } a < a_1 \text{ and } f_5(a) \geq f_4(a) \text{ for } a \geq a_1,$$

$$f_4(a) < f_3(a) \text{ for } a < a_2 \text{ and } f_4(a) \geq f_3(a) \text{ for } a \geq a_2,$$

$$f_3(a) < f_2(a) \text{ for } a < a_3 \text{ and } f_3(a) \geq f_2(a) \text{ for } a \geq a_3,$$

$$f_2(a) < f_1(a) \text{ for } a < a_3 \text{ and } f_2(a) \geq f_1(a) \text{ for } a \geq a_3.$$

This completes the proof of Proposition 3.2.

As $12\tan\frac{\pi}{6}a^2 \leq 8a^2\tan\frac{\pi}{4},\ 5a^2\tan\frac{\pi}{5},\ 10a^2\tan\frac{\pi}{5}$, and $\frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} \leq \frac{21}{4}\tan\frac{\pi}{7}$, $\frac{7}{2}\tan\frac{\pi}{7} + \frac{3}{2}\tan\frac{\pi}{6}$, $\frac{3}{4}\tan\frac{\pi}{6}$, we can easily get the following corollary, which is Lemma 3.1.

COROLLARY **4.3** *Suppose a Euclidean torus T contains three disks D_1, D_2, D_3 with radius $1/2$ and two disks D_4, D_5 with radius $0.11 < a \leq 0.5$ such that $int(D_i) \cap int(D_j) = \emptyset$ for $i \neq j$. Then the following inequality holds;*

$$\operatorname{area}(T) \geq \frac{7}{2}\tan\frac{\pi}{7} + 2\tan\frac{\pi}{8} + 12\tan\frac{\pi}{6}a^2.$$

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