COMMENSURABILITY OF V2050(4,1) AND V3404(1,3)

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Abstract. Commensurable hyperbolic 3-manifolds have the same invariant trace field and invariant quaternion algebra. D. Coulson, O. Goodman, C. Hodgson, and W. Neumann showed that two closed hyperbolic manifolds v2050(4,1) and v3404(1,3) in the closed census have the same invariant trace field and invariant quaternion algebra. In this paper, we show that v2050(4,1) and v3404(1,3) are incommensurable by using the ratio of their volumes.

1. Introduction

Two hyperbolic 3-manifolds are commensurable if they have a common cover, of finite degree. Commensurable manifolds have the same invariant trace field and invariant quaternion algebra. In [1], D. Coulson, O. Goodman, C. Hodgson, and W. Neumann showed that two closed hyperbolic manifolds v2050(4,1) and v3404(1,3) in the closed census have the same invariant trace field and invariant quaternion algebra. They proved these manifolds are incommensurable by using Borel regulator. In this paper, we show that v2050(4,1) and v3404(1,3) are incommensurable by using the ratio of their volumes.

2. Preliminaries

In [3], T. Marshall and G. Martin showed following theorem.

Theorem 1. *Let Γ be a Kleinian group. Then either*
$vol(\mathbb{H}^3/\Gamma) = vol(\mathbb{H}^3/\Gamma_0) = 0.0390...$ *and* $\Gamma = \Gamma_0$*, or*
$vol(\mathbb{H}^3/\Gamma) = vol(\mathbb{H}^3/\Gamma_1) = 0.0408...$ *and* $\Gamma = \Gamma_1$*, or*
$vol(\mathbb{H}^3/\Gamma) > 0.041$.

They also proved that Γ_0 and Γ_1 are arithmetic. We have the following corollary

Corollary 1. *The volume of non-arithmetic orientable closed hyperbolic orbifold is larger than* 0.041.

3. Proof of Main Theorem

G. Margulis proved that the commensurability class of a non-arithmetic finite-volume hyperbolic orbifold has a minimal element [2]. In [1], D. Coulson, O. Goodman, C. Hodgson, and W. Neumann showed that v2050(4,1) and v3404(1,3) are non-arithmetic. The symmetry group of v2050(4,1) and v3404(1,3) is D_2. Put M_1 =v2050(4,1)$/D_2$ and M_2 =v3404(1,3)$/D_2$. They cover a common orientable orbifold Q. Let $P_i : M_i \to Q$ be an n_i-fold covering map. Then we get $vol(M_i) = n_i vol(Q)$ $(i = 1,\ 2)$.

n_1	14	15	16	17	18
$1.28037328\cdots\times n_1$	17.92523	19.20560	20.48597	21.76635	23.04672

19	20	21	22	23	24	25	26
24.32709	25.60747	26.88784	28.16821	29.44859	30.72896	32.00933	33.28971

Table 1

$$\begin{aligned}\frac{vol(M_2)}{vol(M_1)} &= \frac{\frac{vol(M_2)}{vol(Q)}}{\frac{vol(M_1)}{vol(Q)}}\\ &= \frac{n_2}{n_1}.\end{aligned}$$

As $vol(M_1) = 1.0991682\ldots$ and $vol(M_2) = 1.407345594\ldots$, we have

$$\frac{n_2}{n_1} = 1.28037328\ldots. \tag{1}$$

By Proposition 1, $vol(Q) > 0.041$. Thus

$$n_1 = \frac{vol(M_1)}{vol(Q)} < 26.8.$$

Since n_1 is the degree of the covering map p_1, n_1 is a positive integer. $n_1 \in \{1, \cdots, 26\}$. n_2 is also a positive integer. By (1), $n_2 = 1.28037328\cdots\times n_1$. If $1.28037328\cdots\times n_1$ is a integer, $1.28037328\cdots\times 2n_1$ is also a integer. We can check $1.28037328\cdots\times n_1$ is not a integer for $n_1 \in \{14, \cdots, 26\}$. (see Table 1.)

Hence M_1 is incommensurable to M_2.

References

[1] D. Coulson, O. Goodman, C. Hodgson, W. Neumann Computing arithmetic invariants of 3-manifolds Experiment. Math. Volume 9, Issue 1 (2000), 127–152.

[2] G. Margulis. Discrete Subgroups of Semi-simple Lie Groups, Ergeb. der Math. 17, Springer-Verlag (1989).

[3] T. Marshall, G. Martin, Minimal co-volume hyperbolic lattices, II: Simple torsion in a Kleinian group, Annals of Mathematics (2012) 2(176),261–301

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